Facts about Trigonometric Integrals involving Sine and/or Cosine	**Explanation**
Power Notation for Trig. Functions	We usually use the shorthand notation of $sin^m(x)$ to mean $(sin(x))^m$. Note that $sin^{-1}(x)$ denotes the inverse of the sine function and not $(sin(x))^{-1}$. The above two notation rules for sine also hold for the other trigonometric functions.
$sin^2(\theta) + cos^2(\theta) = 1$	Pythagorean Identity used to solve Trigonometric Integrals involving odd powers of sine or cosine.
$sin^2(\theta) = \frac{1 - cos(2\theta)}{2}$	Half-angle (power reducing) formula for sine used to solve Trigonometric Integrals involving even powers of sine.
$cos^2(\theta) = \frac{1 + cos(2\theta)}{2}$	Half-angle (power reducing) formula for cosine used to solve Trigonometric Integrals involving even powers of cosine.
Rules to Integrate Products of Powers of Sine and Cosine (These rules also work when there is only $sin(x)$ and $cos(x)$ in the integrand. That is, m and n can be zero) $\int sin^m(x)cos^n(x)\ dx$	Case 1: If m and n are both odd, then chose only one function (either sine or cosine) to "break one off" and then use the Pythagorean Identity on the remaining even power function. Ignore the power of the other function. Case 2: If m and n do not have the same parity (one is even and the either is odd), then choose the function with an odd power to "break one off" and then use the Pythagorean Identity on the remaining even power function. Ignore the original even power function. Case 3: If m and n are both even, then use the half-angle identities on both sine and cosine.

1. Evaluate $\int sin^3(x)\ dx$.

Solution:

Let $u = cos(x)$, so $du = -sin(x)\ dx$.

$$\begin{aligned}\int sin^3(x)\ dx &= \int sin^2(x)sin(x)\ dx \\ &= \int (1 - cos^2(x))sin(x)\ dx \\ &= \int (1-u^2)(-du) \\ &= \int -1 + u^2\ du \\ &= -u + \frac{u^3}{3} + C \\ &= -cos(x) + \frac{cos^3(x)}{3} + C\end{aligned}$$

2. Evaluate $\int sin^6(x)cos^3(x)\ dx$.

Solution:

Let $u = sin(x)$, so $du = cos(x)\ dx$.

$$\begin{aligned}\int sin^6(x)cos^3(x)\ dx &= \int sin^6(x)(1 - sin^2(x))cos(x)\ dx \\ &= \int u^6(1-u^2)\ du \\ &= \int u^6 - u^8\ du \\ &= \frac{u^7}{7} - \frac{u^9}{9} + C \\ &= \frac{sin^7(x)}{7} - \frac{sin^9(x)}{9} + C\end{aligned}$$

3. Evaluate $\int sin^2(x)cos^4(x)\ dx$.

Solution:

Using $sin^2(x) = \frac{1}{2}(1 - cos(2x))$ and $cos^2(x) = \frac{1}{2}(1 + cos(2x))$, we find

$$\begin{aligned}\int sin^2(x)cos^4(x)\ dx &= \int \frac{1 - cos(2x)}{2} \cdot \left(\frac{1 + cos(2x)}{2}\right)^2\ dx \\ &= \frac{1}{8}\int 1 + cos(2x) - cos^2(2x) - cos^3(2x)\ dx \\ &= \frac{1}{8}\left(\int 1 + cos(2x) - cos^2(2x)\ dx - \int cos^3(2x)\ dx\right),\end{aligned}$$

where the integral has been split to emphasize that the last part is similar to problem 1! Now, using $cos^2(2x) = \frac{1}{2}(1 + cos(4x))$ and $u = sin(2x)$, we find

$$\begin{aligned}\int 1 + cos(2x) - cos^2(2x)\ dx &= \int 1 + cos(2x) - \frac{1}{2} - \frac{1}{2}cos(4x) \\ &= \frac{x}{2} + \frac{sin(2x)}{2} - \frac{sin(4x)}{8} + C\end{aligned}$$

and

$$\begin{aligned}\int cos^3(2x)\ dx\ dx &= \int (1 - sin^2(2x))cos(2x)\ dx \\ &= \frac{1}{2}\int (1 - u^2)\ du \\ &= \frac{u}{2} + \frac{u^3}{6} + C \\ &= \frac{sin(2x)}{2} + \frac{sin^3(2x)}{6} + C\end{aligned}$$

Finally, we find

$$\begin{aligned}\int sin^2(x)cos^4(x)\ dx &= \frac{1}{8}\left(\frac{x}{2} + \frac{sin(2x)}{2} - \frac{sin(4x)}{8} + C - \left(\frac{sin(2x)}{2} + \frac{sin^3(2x)}{6} + D\right)\right) \\ &= \frac{1}{8}\left(\frac{x}{2} - \frac{sin(4x)}{8} - \frac{sin^3(2x)}{6} + C\right) \\ &= \frac{x}{16} - \frac{sin(4x)}{64} - \frac{sin^3(2x)}{48} + C.\end{aligned}$$

Make sure you understand what happened with the constants!

Facts about Trigonometric Integrals involving Tangent and Secant	**Explanation**
$1 + tan^2(x) = sec^2(x)$	Pythagorean Identity used to solve Trigonometric Integrals involving powers of tangent and secant.
Rules to Integrate Products of Powers of Tangent and Secant $$\int tan^m(x)sec^n(x)\ dx$$	Case 1: If n (the power of secant) is even, then break off $sec^2(x)$ and use the Pythagorean Identity ($sec^2(x) = 1 + tan^2(x)$) on the remaining even power of secant. Case 2: If m (the power of tangent) is odd, break off one $sec(x)$ and $tan(x)$ ($sec(x)tan(x)$), then use the Pythagorean Identity ($tan^2(x) = sec^2(x) - 1$) on the remaining even power of tangent.

4. Evaluate $\int tan^6(x)\ sec^4(x)\ dx$.

Solution:

Let $u = tan(x)$, so $du = sec^2(x)$.

$$\begin{aligned}
\int tan^6(x)sec^4(x)\ dx &= \int tan^6(x)sec^2(x)sec^2(x)\ dx \\
&= \int tan^6(x)(1 + tan^2(x))sec^2(x)\ dx \\
&= \int u^6(1 + u^2)\ du \\
&= \frac{u^7}{7} + \frac{u^9}{9} + C \\
&= \frac{tan^7(x)}{7} + \frac{tan^9(x)}{9} + C
\end{aligned}$$

5. Evaluate $\int tan^5(x)sec^9(x)\ dx$.

Solution:

Let $u = sec(x)$, so $du = sec(x)tan(x)dx$.

$$\begin{aligned}
\int tan^5(x)sec^9(x)\ dx &= \int tan^4(x)sec^8(x)sec(x)tan(x)\ dx \\
&= \int (sec^2(x)-1)^2 sec^8(x)sec(x)tan(x)\ dx \\
&= \int \left(u^2-1\right)^2 u^8\ du \\
&= \int (u^4-2u^2+1)u^8\ du \\
&= \int u^{12}-2u^{10}+u^8\ du \\
&= \frac{u^{13}}{13}-\frac{2u^{11}}{11}+\frac{u^9}{9}+C \\
&= \frac{sec^{13}(x)}{13}-\frac{2sec^{11}(x)}{11}+\frac{sec^9(x)}{9}+C
\end{aligned}$$